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BROWNIAN MOTION ON ANOSOV FOLIATIONS AND MANIFOLDS OF NEGATIVE CURVATURE

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Abstract

We study ergodic properties of Anosov foliations. Some rigidity results are obtained, including applications to manifolds of negative curvature, and an integral formula for topological entropy. We also show that the function $c(x)$ in Margulis's asymptotic formula $c(x) = \lim_{R\to\infty} e^{-hR} \cdot S(x, R)$ is almost always not constant. In dimension 2, $c(x)$ is a constant function if and only if the manifold has constant negative curvature. Generally, if the Ledrappier-Patterson-Sullivan measure is a flip invariant, then $c(x)$ is constant. Our entropy formula yields an upper bound of Gromov's simplicial volume in terms of scalar curvature.

0. Introduction

We generalize Lucy Garnett's ergodic theory for C^3 foliations to foliations $\mathcal{F}$ of class $C^3_{\mathcal{F}}$ (for definition see §1.1), and apply it to study the ergodic properties of Anosov foliations. In §1.3 we prove

Theorem 1. *The horocycle foliations* (W^{su} *or* W^{ss}) *of a* C^3*-transitive Anosov system with leafwise Riemannian metric of class* C^3_i ($i = su, ss$) *are uniquely ergodic* (*i.e., they have precisely one harmonic measure*).

Then we generalize the integral formulas in [27] to Anosov foliations to obtain the following rigidity result:

Theorem 2. *For an Anosov system with its unique harmonic measure* w^{ss}, *the following properties are equivalent:*

1° w^{ss} *is an invariant measure of the Anosov system.*

2° J^{ss}_t *is constant along* W^{ss}*-leaves.*

We apply the above theory to the geodesic flow on a compact Riemannian manifold M of negative curvature. We give an explicit description of the harmonic measure w^{ss} as the weak limit of the normalized spherical measure of geodesic balls. This settles a problem raised by Katok. We also derive two formulas for the topological entropy.

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Theorem 3. *Let R be the scalar curvature of M, and R^H the scalar curvature of the horospheres. Let* Ric *be the Ricci curvature of M, and* $\operatorname{tr} U$ *be the mean curvature of the horospheres. Then the topological entropy h satisfies*

1.° $h = \int_{SM} \operatorname{tr} U \, dw^{ss}$,

2.° $h^2 = \int_{SM} (R^H(v) - R(\pi(v)) + \operatorname{Ric}(v)) \, dw^{ss}$,

where $\pi(v)$ is the base point of the vector v.

Using A. Connes' Gauss-Bonnet theorem for foliation we get

Corollary. *For a 3-dimensional closed Riemannian manifold of negative curvature,*

$$h^2 = \int_{SM} (\operatorname{Ric}(v) - R(\pi(v)) \, dw^{ss}(v).$$

In §4, we study the Margulis' asymptotic formula

$$\lim_{R \to \infty} \frac{1}{e^{hR}} S(x, R) = c(x)$$

for the volume of geodesic spheres. Twenty years ago, Margulis [18] obtained this celebrated formula. He commented on the first page of his short paper that $c(x)$ is a positive continuous function. A. Katok conjectured that $c(x)$ is almost always not constant and not smooth. We now prove

Theorem 4. *For any compact manifold M of negative curvature, the following hold:*

1° *$c(x)$ is smooth.*

2° *If $c(x)$ is a constant function, then for each x in $\widetilde{M}$,*

$$h = \int_{\partial \widetilde{M}} \operatorname{tr} U(x, \xi) \, d\mu_x(\xi),$$

where μ_x is the Bowen-Ledrappier-Margulis-Patterson-Sullivan measure at infinity.

Theorem 4 implies particularly

Theorem 5. *If* dim $M = 2$, *then $c(x)$ is a constant function if and only if M has constant negative curvature.*

In §4.3 we discuss the flip-invariance of Patterson-Sullivan measure, and §4.4 contains some applications to Gromov's simplicial volume.

1. Preliminaries and Anosov foliation

1.1. Ergodic properties of foliations. In this section we review the main results in [4] and generalize them to foliations $\mathcal{F}$ of class $C^3_{\mathcal{F}}$.

Let $\mathscr{F}$ be any foliation on a compact manifold M equipped with a Riemannian metric on its tangent bundle. We assume that both $\mathscr{F}$ and the Riemannian metric on its tangent bundle are of class C^3. Each leaf L of the foliation inherits a C^3 Riemannian structure making it into a connected C^3 Riemannian manifold. The induced geometries on the leaves are uniformly bounded because M is compact. Thus each leaf L is complete for diffusion (i.e., the integral of the heat kernel over the whole space equals one).

We have a Laplace-Beltrami operator Δ^L on each leaf L. The measure on the leaf L induced by the Riemannian metric is denoted by dx. Let $P_t(x, y)$ be the heat kernel of the operator Δ^L. Then there is a one-parameter semigroup of operators D_t corresponding to the diffusion of heat in the leaf directions:

$$D_t f(x) = \int_L f(y) P_t(x, y)\, dy,$$

where f is a global function $f : M \to R$. If m is a measure on M, the measure diffused along the leaves of the foliation $D(t)m$ is defined by

$$\int_M f\, d(D(t)m) = \int_M D_t f\, dm.$$

The set of probability measures on a compact finite-dimensional foliated manifold M is a nonempty convex set. The leaf diffusion operator $D(t)$ is a continuous affine mapping, and any fixed point will be diffusion invariant for the time t. The Markov-Kakutani fixed point theorem insures that a fixed point exists for all times.

Definition. (i) A probability measure on M is said to be diffusion invariant if the integral of f with respect to that measure equals the integral of $D_t f$ with respect to the measure for any continuous function f.

(ii) A diffusion invariant measure is said to be ergodic if the manifold M cannot be split into two disjoint measurable leaf saturated sets with intermediate measure.

(iii) A probability measure m on M is harmonic if $\int_M \Delta^L f\, dm = 0$, where f is any bounded measurable function on M, which is smooth in the leaf direction, and Δ^L denotes the Laplacian in the leaf direction.

Let E be any flow box of the foliation $\mathscr{F}$. The quotient of such an E by the local $\mathscr{F}$-leaves is called the quotient transversal $I = I(E)$. If $P : E \to I$ is the projection, then by the classical measure theory, any measure m on E may be disintegrated uniquely into the projected measure ν on the transversal I and a system of measures $\sigma(s)$ on the leaf slices $p^{-1}(s) = E(s)$ for each s in I. These measures satisfy the following conditions:

(i) $\sigma(s)$ is a probability measure on $E(s)$.
(ii) If S is a measurable subset of I, then $\nu(S) = m(p^{-1}(S))$.
(iii) If f is m-integrable and $\operatorname{supp}(f) \subset E$, then

$$\int f(x)\,dm(x) = \int f(y)d\sigma(s)(y)\,d\nu(s).$$

The following theorem is due to Lucy Garnett.

Theorem ([4]). *Let M be a compact foliated manifold with C^3-foliation $\mathscr{F}$ and a C^3-Riemannian metric on the tangent bundle of $\mathscr{F}$. Let m be any probability measure on M, then the following conditions are equivalent:*

(i) *m is diffusion invariant, i.e., $D(t)m = m$ for all t.*
(ii) *m is harmonic, i.e., $\int_M \Delta^L \varphi\,dm = 0$ for any bounded measurable function on M, which is smooth in the leaf direction.*
(iii) *For any flow box E of the foliation $\mathscr{F}$ and ν, almost all s (see the above construction), $\sigma(s)$ is a harmonic function times the Riemannian measure restricted to $E(s)$.*

Recall that a holonomy invariant measure of the foliation $\mathscr{F}$ is a family of measures defined on each transversal of the foliation $\mathscr{F}$, which is invariant under all the canonical homeomorphisms of the holonomy pseudogroup (see [19]). Given any transverse invariant measure, a global measure may be formed by locally integrating the Riemannian leaf measures with respect to the transverse invariant measure. Such a measure is said to be completely invariant. Obviously, any such measure disintegrates locally to a constant function times the Riemannian leaf measure and thus, by Theorem 1, is an harmonic measure for the foliation $\mathscr{F}$.

If L is a leaf of $\mathscr{F}$, let $x \in L$ and let $B(x, R)$ denote the ball in L of radius R around x under the leaf Riemannian metric. Define the growth function of $\mathscr{F}$ at x by $G_x(R) = \operatorname{vol} B(x, R)$ where vol denotes the Riemannian volume on the leaf L. L is said to have exponential growth if

$$\varliminf_{R\to\infty} \frac{1}{R} \log G_x(R) > 0,$$

and nonexponential growth otherwise.

Every foliation admits a nontrivial harmonic measure. But the following theorem tells us that there are many foliations which have no holonomy invariant measure at all.

Theorem ([19]). *For a codimension-one foliation* $\mathscr{F}$ *of class* C^1 *of a compact manifold* M *the following are equivalent.*

(i) $\mathscr{F}$ *has a leaf with nonexponential growth.*
(ii) $\mathscr{F}$ *has a leaf with polynomial growth.*
(iii) $\mathscr{F}$ *has a nontrivial holonomy invariant measure.*

For arbitrary codimension, we have

Theorem ([19]). *Let* $\mathscr{F}$ *be a foliation of class* C^1 *of a compact manifold* M. *If* L *is a leaf of* $\mathscr{F}$ *having nonexponential growth, then there exists a nontrivial holonomy invariant for* $\mathscr{F}$, *which is finite on compact sets and has support contained in the closure of* L.

As a direct corollary of Yosida's ergodic theorem for Markov processes (see [25]), one has the following.

The foliation ergodic theorem ([4]). *Let* M *be a harmonic probability measure. For any* m*-integrable function* f *there exists an* m*-integrable function* $\tilde{f}$ *which is constant along the leaves and satisfies the following*:

(i) $\tilde{f}(x) = \lim_{t\to\infty} \frac{1}{T}\int_0^T D_t f(x)\,dt$ *for* m *almost all* x.
(ii) $\int \tilde{f}(x)\,dm = \int f(x)\,dm$.
(iii) *If* m *is ergodic, then* $\tilde{f} = \int f(x)\,dm(x)$.

Let us denote by $\{w_t\}$ the set of Brownian paths lying on the leaves of the foliation $\mathscr{F}$ (induced by the Riemannian metric on each leaf). Then we have another interpretation of the foliation ergodic theorem.

The leaf path ergodic theorem ([4]). *Let* m *be any harmonic probability measure. For any* m*-integrable function* f *on* M, *the limit*

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T f(w_t)\,dt$$

exists for m*-a.e.* x *and almost any path* w (*in the sense of Wiener measure*) *starting at* x *and lying on the leaf on* x. *This limit is constant on leaves and equals the leaf diffused time average of* f.

Finally, we have the Kryloff-Bogoluboff theory of harmonic measures.

Theorem. *There is a leaf saturated measurable set* R *in* M *having the following properties:*

(i) *For any* $x \in R$ *the diffused Dirac measure* $\tilde{\delta}_x$ *exists, is ergodic and contains* x *in its support where* $\tilde{\delta}_x$ *is defined by* $\int f\,d\tilde{\delta}_x = \tilde{f}(x)$ *for any continuous function* $f : M \to R$.
(ii) *Any two points on the same leaf in* R *have the same diffused Dirac measures.*
(iii) R *has full probability* (*i.e.,* $u(R) = 1$ *for any harmonic probability measure*).

If one checks carefully all the steps of [4] (particularly the proofs of Facts 1–4 on pp. 289–292), one easily sees that all the results in [4] are true for foliations with C^3-leaves and C^3 Riemannian metric on each leaf whose 3-jets depend continuously on the points in M.

Let us be more specific about the regularity requirements for the foliations and the leafwise Riemannian metrics. We say that a foliation $\mathscr{F}$ has C^k leaves and C^k Riemannian metric on each leaf whose k-jets depend continuously on the points in M if for each point in M there is a local parametrization of the $\mathscr{F}$ foliation $\varphi : U \times V \to M$ (where $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^c$ are open sets, $d = \dim\mathscr{F}$ and $c = \operatorname{codim}\mathscr{F}$), such that the following hold:

(i) φ is a homeomorphism from $U \times V$ to an open set in M.

(ii) For each $y \in V$, $\varphi_y : U \to M$ given by $\varphi_y(x) = \varphi(x, y)$ is a C^k immersion whose image is an open subset of a leaf of the W^u-foliation, and, moreover, for any $1 \leq \alpha \leq k$, $\partial^\alpha\varphi/\partial x^\alpha$ is continuous on $U \times V$.

(iii) For each $y \in V$, the pull-back of the Riemannian metric $g^{\mathscr{F}}$ on the leaf $\varphi_y(U)$ is a Riemannian metric on U satisfying $\varphi_y^*(g) = g_{ij}dx^i \wedge dx^j$, $0 \leq i, j \leq d$, such that for each $1 \leq \alpha \leq k$, $\partial^\alpha g_{ij}/\partial x^\alpha$ is continuous on $Y \times V$.

Such a foliation is said to be of class $C^k_{\mathscr{F}}$. Such a Riemannian metric on leaves is also said to be of class $C^k_{\mathscr{F}}$. We will say that a function $\psi : M \to R$ is of class $C^j_{\mathscr{F}}$, $0 \leq j \leq k$, if $\psi \circ \varphi : U \times V \to R$ has derivatives of orders $1 \leq \alpha \leq j$ with respect to the arguments in U and that these derivatives are continuous.

1.2. Entropy properties of foliations. We review the main results in [10]. Our setting is a $C^3_{\mathscr{F}}$ foliation $\mathscr{F}$ on a compact manifold M and leafwise Riemannian metrics of class $C^3_{\mathscr{F}}$. Let $P_t(x, y)$ be the heat kernel of the leaf L_x. For any harmonic ergodic measure m, the densities φ of the conditional measures m on the $\mathscr{F}$-leaves are uniquely determined up to a scalar multiplication. Then we have a biased random motion (the φ-process) corresponding to the second order operator $\Delta^L + 2\nabla^L \log\varphi$, where ∇^L is the gradient operator on L. The transition probability densities for this φ-process are

$$P_t'(x, y) = P_t(x, y)\varphi(y)/\varphi(x).$$

Theorem (Kaimanovich, [10]). *For m-a.e. x in M the following limits exist* (*not depending on* x):

$$h(\mathscr{F}, m) = -\lim_{t\to\infty} \frac{1}{t} \int P_t(x, y) \log P_t(x, y)\, dy,$$
$$h'(\mathscr{F}, m) = -\lim_{t\to\infty} \frac{1}{t} \int P_t'(x, y) \log P_t'(x, y)\, dy,$$

where $h(\mathscr{F}, m)$ *is called the entropy of Brownian motion on the foliation* $\mathscr{F}$ *with respect to* m.

We list the following facts in [10] by recalling that a Riemannian manifold M is said to be Liouvillian if there are no nonconstant bounded harmonic functions on M:

1° $h(\mathscr{F}, m) = h'(\mathscr{F}, m) + \int_M \|\nabla^L \log\varphi\|^2 dm$.

2° $h(\mathscr{F}, m) = 0$ (or $h'(\mathscr{F}, m) = 0$) if and only if m-a.e. leaf is Liouvillian.

3° If $h(\mathscr{F}, m) > 0$ (or $h'(\mathscr{F}, m) > 0$), then for m-a.e. leaves of the foliation, the space of bounded harmonic functions is infinite-dimensional.

4° Every harmonic ergodic measure with almost all leaves Liouvillian leaves is completely invariant.

5° If m-a.e. leaf of the foliation has subexponential growth, then m-a.e. leaf is Liouvillian.

Note that if the foliation $\mathscr{F}$ is trivial (i.e., the manifold M itself), then for all $x \in M$, the following limit exists:

$$h(M) = -\lim_{t\to\infty} \frac{1}{t} \int_M P_t(x, y) \log P_t(x, y)\, dm(y).$$

Moreover, the following hold:

1° M is Liouvillian if and only if $h(M) = 0$.

2° If M has subexponential growth, then M is Liouvillian.

1.3. Harmonic measures for Anosov foliations. In this section we consider a transitive Anosov flow g_t on a closed manifold M (or a transitive Anosov diffeomorphism f on M). We denote by W^{su} (resp. W^{ss}) the strong unstable (resp. strong stable) foliation of the Anosov flow g_t or the Anosov diffeomorphism f. These foliations are also known as horospheric foliations. For Anosov flows, we also get the weak stable foliations W^s (resp. weak unstable foliations W^u):

$$W^s(x) = \bigcup_{t\in\mathbb{R}} W^{ss}(g_t x) \quad (\text{ resp. } W^u = \bigcup_{t\in\mathbb{R}} W^{su}(g_t x)).$$

All these Anosov foliations are always Hölder continuous but may fail to be C^1, even if the flow (or the diffeomorphism) itself is C^∞. But it is well-known that if the Anosov flow g^t (or the Anosov diffeomorphism f) is C^k ($k \geq 2$ is any integer or ∞), then each W^i-leaf ($i = s, u, ss, su$) is a C^k immersed manifold (see, e.g., [21]). Moreover, the four foliations have C^k leaves whose k-jet continuously depend on the point. As remarked by R. De La Llave, J. M. Marco, and R. Moriyon in their fundamental work ([17, pp. 578]), their regularity results for the Livsic cohomology equation can also be stated in terms of functions and flows or diffeomorphisms of class C^k by using the Sobolev embedding theorem.

Let us consider any Riemannian metric g^i defined on the W^i-foliation ($i = s, u, ss, su$) of class C_i^3 (note that for any C^3-Riemannian metric on M, the induced Riemannian metrics g^i ($i = ss, su, s, u$) are of class C_i^3). By Proposition 1.1, each of these foliations has nontrivial harmonic measures.

Theorem 1.1. *The horocycle foliations (W^{su} or W^{ss}) of a C^3-transitive Anosov system with leafwise Riemannian metric of class C_i^3 ($i = su, ss$) are uniquely ergodic in the sense that there is precisely one harmonic probability measure.*

Proof. We consider only the W^{su} foliation (the W^{ss} foliation can be treated similarly). According to a result of D. Sullivan and R. Williams [24], the leaves of any strong Anosov foliations have polynomial growth. By Fact 4° of 1.2, any W^{su}-harmonic ergodic measure m is completely invariant. Now Theorem 3.1 follows from Bowen and Marcus' result that the horocycle foliations have unique holonomy invariant measure [1].

We denote this measure by w^{su}. It has a local description as the product of Lebesgue measure m^{su} on W^{su} and Bowen-Margulis measure μ^{ss} on W^{ss} (in the diffeomorphism situation) or μ^s on W^s (in the flow case):

$$dw^{su} = m^{su} \times \mu^{ss} \quad (\text{or } m^{su} \times \mu^s \text{ in the flow case}).$$

Note that the unique harmonic measure w^{ss} of the W^{ss}-foliation has a similar description:

$$dw^{ss} = m^{ss} \times \mu^{su} \quad (\text{or } m^{ss} \times \mu^u \text{ in the flow case}).$$

Recall that a flow φ_t on a compact metric space X is uniquely ergodic if any only if the sequence

$$f_T(x) \triangleq \frac{1}{T} \int_0^T f(\varphi_t(x))\, dt$$

converges uniformly for any continuous function f on X:

$$f_T(x) \to \int f(x)\, d\mu(x) \qquad (T \to \infty),$$

where μ is the unique ergodic measure. We have an analogous result for the W^{ss} (or W^{su}) foliation.

Theorem 1.2. *For any continuous function f on M,*

$$\frac{1}{m^{ss}B_x^{ss}(R)}\int_{B_x^{ss}(R)} f(y)\,dm^{ss}(y) \longrightarrow \int f(x)\,dw^{ss}(x)$$

uniformly on M, where $B_x^{ss}(R)$ is the ball on $W^{ss}(x)$ with respect to the g^{ss} metric.

Proof. By the arguments of Sullivan [22] and Plante [19], for any foliation $\mathscr{F}$ with subexponential growth, the normalized measures

$$\frac{1}{m^{\mathscr{F}}B_x^{\mathscr{F}}(R)}\int_{B_x^{\mathscr{F}}(R)} \bullet\, dm^{\mathscr{F}}(y)$$

is weakly coverging to some harmonic measure of the foliation. Our theorem follows from the following facts:

(i) W^{ss} is a foliation with polynomial growth.

(ii) The harmonic measure on W^{ss} is unique.

As is remarked in [1], the weak stable (or weak unstable) foliation W^s (or W^u) for an Anosov flow has no completely invariant measure. However, they have at least one nontrivial harmonic measure.

Conjecture. *The weak stable (or unstable) Anosov foliations are uniquely ergodic in the sense that they have a unique harmonic measure.*

This is true in the special case of geodesic flows on manifolds with negative curvature (see §2).

1.4. Integral formula and rigidity. We continue to use the assumptions and notation of §3. For a C^3 Anosov flow g_t or an Anosov diffeomorphism f with leafwise Riemannian metrics g^i on W^i of class C_i^3 ($i = ss, su$), we define

$$\varphi^u(x) = -\frac{d}{dt}\bigg|_{t=0} \log J_t^{su}(x) \qquad (J_t^{su}(x) \triangleq \det dg^t|_{W^{su}(x)}),$$

$$\varphi^s(x) = \frac{d}{dt}\bigg|_{t=0} \log J_t^{ss}(x) \qquad (J_t^{ss}(x) \triangleq \det dg^t|_{W^{ss}(x)}),$$

for a flow. For a diffeomorphism, we define

$$\varphi^u(x) = -\log J_1^{su}(x) \qquad (J_t^{su}(x) \triangleq \det\, df^t|_{W^{su}(x)}),$$

$$\varphi^s(x) = \log J_1^{ss}(x) \qquad (J_t^{ss}(x) \triangleq \det\, df^t|_{W^{ss}(x)}).$$

It is well known that φ^u (resp. φ^s) is Hölder continuous and has a unique equilibrium state m^+ (resp. m^-) which is an invariant ergodic measure of the Anosov system [2]. It is also uniquely determined by the fact that it disintegrates into absolutely continuous measures along the W^{su} (resp. W^{ss}) leaves.

If we denote by ρ^{su} (resp. ρ^{ss}) the local density of conditional measures of m^+ (resp. m^-) with respect to the Riemannian volume m^{su} (resp. m^{ss}), then ρ^{su} (resp. ρ^{ss}) is of class C^3_{su} (resp. C^3_{ss}) (see [17], [16], [27]), and $\nabla^{su}\log\rho^{su}$ (resp. $\nabla^{ss}\log\rho^{ss}$) is a continuous vector field on M of class C^2_{su} (resp. C^2_{ss}).

Theorem 1 in [27] can be generalized to an arbitrary Anosov system.

Theorem 1.3. (i) *For any C^2_{su} function φ on M, we have*

$$\int_M (\Delta^{su}\varphi + \langle\nabla^{su}\varphi, \nabla^{su}\log\rho^{su}\rangle)\, dm^+ = 0.$$

(ii) *For any C^2_{ss} function φ on M, we have*

$$\int_M (\Delta^{ss}\varphi + \langle\nabla^{ss}\varphi, \nabla^{ss}\log\rho^{ss}\rangle)\, dm^- = 0.$$

We have the following rigidity results.

Theorem 1.4. *For an Anosov system (flow or diffeomorphism), the following properties are equivalent:*

(a) *The measure m^+ (resp. m^-) and the measure w^{su} (resp. w^{ss}) coincide.*

(b) *w^{su} (resp. w^{ss}) is an invariant measure of the Anosov system.*

(c) *$J^{su}_t(x)$ (resp. $J^{ss}_t(x)$) is constant along W^{su} (resp. W^{ss}) leaves.*

Proof. (a) $\Longrightarrow$ (b) is obvious.

(b) $\Longrightarrow$ (a). The measure w^{su} is invariant and absolutely continuous along the W^{su}-foliation. By the uniqueness of the Bowen-Sinai-Margulis measure, we have $w^{su} = m^+$.

(a) $\Longrightarrow$ (c). Consider all those φ with compact support in a local W^{su}-flow box P. Then $m^+ = w^{su}$ implies

$$0 = \int \Delta^{su}\varphi\, dm^+ = \int_P \int_{W^{su}_{\mathrm{loc}}(x)\cap P} (\Delta^{su}\varphi)\rho^{su}\, dm^{su}(y)$$
$$= \int_P \int_{W^{su}_{\mathrm{loc}}(x)\cap P} \varphi(\Delta^{su}\rho^{su})\, dm^{su}(y).$$

By the arbitrariness of φ, $\Delta^{su}\rho^{su} = 0$. On the other hand, according to [16],

$$\frac{\rho^{su}(y)}{\rho^{su}(x)} = \prod_{i=1}^{\infty} \frac{J^{su}_1(f^{-i}x)}{J^{su}_1(f^{-i}y)}.$$

Thus ρ^{su} is a bounded harmonic function along each W^{su}-leaf, and must be constant along each W^{su}-leaf.

(c) $\Longrightarrow$ (b). Obvious by the description of the Bowen-Margulis measure.

2. Applications to manifolds of negative curvature

Let M be a closed C^∞ Riemannian manifold of negative curvature, and $\widetilde{M}$ be its universal covering. The geodesic flow g^t on the unit tangent bundle SM is Anosov. We list the following notation:

- $\pi : S\widetilde{M} \to \widetilde{M}$ is the canonical projection.
- $\partial\widetilde{M}$: the ideal boundary of $\widetilde{M}$.
- $v(t) = \pi(g^t v)$ is the geodesic in $\widetilde{M}$ with initial velocity v.
- $P : S\widetilde{M} \to \partial\widetilde{M}$ is the projection $P(v) = v(\infty) \triangleq \lim_{t\to\infty} v(t) \in \partial\widetilde{M}$.
- $P_x : S_x\widetilde{M} \to \partial\widetilde{M}$ is the restriction of P to $S_x\widetilde{M}$.
- (x, ξ): the vector v in S_xM such that $v(\infty) = \xi$.
- ρ_v: the Busemann function at $v(\infty)$ such that $\rho_v(v(0)) = 0$.
- H_v: the horosphere at $v(\infty) \in \partial\widetilde{M}$ passing through $v(0) \in \widetilde{M}$.
- μ, ν, m are the Bowen-Margulis, the harmonic, and the Liouville measure of g^t, respectively.

The canonical projection $\pi: S\widetilde{M} \to \widetilde{M}$ maps $W^i(v)$ $(i = s, u)$ diffeomorphically onto $\widetilde{M}$. Thus the Riemannian metric on $\widetilde{M}$ lifts to a Riemannian metric g^i on $W^i(v)$, which induces a Riemannian volume $m^i (i = s, u)$. π also maps $W^i(v)$ $(i = su, ss)$ diffeomorphically to horospheres on $\widetilde{M}$. The induced Riemannian metrics on horospheres lift to Riemannian metrics g^{su} (or g^{ss}) on $W^{su}(v)$ (or $W^{ss}(v)$), which induces Riemannian volumes m^{su} (or m^{ss}).

Note that all the foliations W^i and the metrics g^i are of class C_i^∞, $i = s, u, ss, su$ (see for example [27]). Thus all the results in §1 apply here, and by summarizing we have

Theorem 2.1. *The W^{su} foliation has a unique harmonic measure w^{su} and, locally, $dw^{su} = dm^{su} \times d\mu^s$. Moreover, the following properties are equivalent*:

(a) $w^{su} = m$.
(b) $w^{su} = \mu$.
(c) $w^{su} = \nu$.
(d) w^{su} *is g^t-invariant.*

(e) *Horospheres in $\widetilde{M}$ have constant mean curvature.*

A similar result can be stated for the W^{ss} foliation and the measure w^{ss}.

The purpose of this section is to prove that the weak stable or unstable foliations are also uniquely ergodic. But first let us recall some basic facts.

Let $\Omega = C(R_+, M)$ be the space of continuous paths in M, and $\{P_x, x \in M\}$ the family of probability measures on Ω which describe the Brownian motion on M. Let $\widetilde{\Omega} = C(R_+, \widetilde{M})$ be the space of continuous paths in $\widetilde{M}$, and $\Pi : \widetilde{M} \to M$ the covering map. For each $x \in \widetilde{M}$ and $w \in \Omega$ such that $w(0) = \Pi(x)$, there is a unique path $\widetilde{w} \in \widetilde{\Omega}$ such that $\Pi(\widetilde{w}(t)) = w(t)$ for all $t \geq 0$. We denote the polar coordinates about x of the path $\widetilde{w}(t)$ by $(r(w, t), \theta(w, t))$.

1° For all x in $\widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$, $\theta(w, t) \overset{t\to\infty}{\longrightarrow} \theta(w, \infty) \in \partial\widetilde{M}$ ([20]). We denote by ν_x the hitting probability measure on $\partial\widetilde{M}$ of Brownian motion starting at x, and

$$\frac{d\nu_y}{d\nu_x}(\xi) = k(x, y, \xi)$$

for all $x, y \in \widetilde{M}$ and almost all $\xi \in \widetilde{M}$. $k(x, y, \xi)$ is called the Poisson kernel.

2° [14] For all $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} d(\widetilde{w}(t), \ (r(w, t), \ \theta(w, \infty))) = 0.$$

3° [9] For all $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} r(w, t) = \alpha.$$

4° [9] $\alpha = \int_M dm(x) \int_{\partial\widetilde{M}} \operatorname{tr} U(x, \xi)\, d\nu_x(\xi)$,
where dm is the Riemannian volume on M, and $U(x, \xi)$ is the second fundamental form at x of the horosphere $H(x, \xi)$. $\operatorname{tr} U(x, \xi) = -\frac{d}{dt}|_{t=0} \log J_t^{ss}(x, \xi)$ is the mean curvature of $H(x, \xi)$ at x.

5° [9] For every $x \in \widetilde{M}$ and $P_{\pi x}$-a.e. $w \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \log P(t, x, \widetilde{w}(t)) = -\beta.$$

6° [9] $\beta = \int_M dm(x) \int_{S_x M} \|\nabla \log k\|^2 \, d\nu_x(P_x(v)) = h_\nu \alpha$, where h_ν is the metric entropy of ν.

Note the measure $dm(x) \times d\nu_x(\xi)$ which appeared in 4°, 6°. We prove that this is the only harmonic measure of the W^s foliation.

Theorem 2.2. *The weak stable foliation of the geodesic flow has a unique harmonic measure* w^s, *described by* $\int_{SM} \cdot dw^s = \int_M dm(x) \int_{S_xM} \cdot d\nu_x(v)$.

Our proof is inspired by Garnett's proof of a special case where M is a surface of constant curvature -1.

Proof. Let us consider a W^s-flow box of the form

$$E = \{(x, \xi) | x \in B, \ \xi \in U\},$$

where B is a ball in M centered at a point x_0, and U is a open set in $\partial\widetilde{M}$. For any continuous function f with compact support in E, we have

$$\begin{aligned}\int_E f\,dw^s &= \int_B dm(x) \int_{\partial\widetilde{M}} f\,d\nu_x(\xi) \\ &= \int_B \int_U f(x, \xi)\, k(x_0, x, \xi)\, dm(x)\, d\nu_{x_0}(\xi) \\ &= \int_U d\nu_{x_0}(\xi) \int_B f(x, \xi)\, k(x_0, x, \xi) dm(x),\end{aligned}$$

which means that dw^s disintegrates locally into the harmonic function $k(x_0, x, \xi)$ times Riemannian volume of the W^s leaf. Thus w^s is a harmonic measure of the W^s foliation.

Given any other ergodic harmonic measure σ and any continuous function f on SM, by the leaf path ergodic theorem, for σ-a.e. leaf $W^s(x_0, \xi)$, for all points $(y, \xi) \in W^s(x_0, \xi)$ and P_y almost any path w starting at y, we have

$$\int_{SM} f\,d\sigma = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(\widetilde{w}(t), \xi)\,dt.$$

Given any other w^s-typical leaf $W^s(x_0, \eta)$, for P_y almost any path w starting at y we have

$$\int_{SM} f\,dw^s = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(\widetilde{w}(t), \eta)\,dt.$$

Consider a typical path $\widetilde{w}$ starting at y such that $\widetilde{w}(t) \to e \in \partial\widetilde{M}$, $e \neq \xi$, $e \neq \eta$. By comparison with manifolds of constant negative curvature, it is easy to see that

$$d_{\widetilde{w}(t)}((\widetilde{w}(t), \xi), \ (\widetilde{w}(t), \eta)) \to 0 \qquad (t \to \infty),$$

where $d_{\widetilde{w}(t)}$ is the induced Riemannian metric on $S_{\widetilde{w}(t)}\widetilde{M}$. Thus

$$\lim_{t\to\infty}\frac{1}{T}\int_0^T [f(\widetilde{w}(t), \xi) - f(\widetilde{w}(t), \eta)]dt = 0,$$

and $\int_{SM} f\,d\sigma = \int_{SM} f\,dw^s$, which proves the uniqueness of harmonic measure of the weak stable foliation.

Note the following:

(i) For any function φ on SM of class C_s^2, we have

$$\begin{aligned}
0 &= \int_M dm(x)\Delta\left(\int_{S_xM}\varphi\,d\nu_x\right)\\
&= \int_M dm(x)\Delta|_{y=x}\left(\int_{S_xM}\varphi k(x, y, \xi)\,d\nu_x(\xi)\right)\\
&= \int_{SM}(\Delta^s\varphi + 2\langle\nabla^s\varphi, \nabla^s\log k\rangle)\,dm(x)\,d\nu_x.
\end{aligned}$$

(ii) We also have

$$\begin{aligned}
0 &= \int_M dm(x)\mathrm{div}\left(\int_{S_xM}\nabla^s\varphi\,d\nu_x\right)\\
&= \int_M dm(x)\mathrm{div}|_{y=x}\left(\int_{S_xM}\nabla^s\varphi\cdot k(x, y, \xi)\,d\nu_x(\xi)\right)\\
&= \int_{SM}(\Delta^s\varphi + \langle\nabla^s\varphi, \nabla^s\log k\rangle)\,dm(x)\,d\nu_x.
\end{aligned}$$

Combining (i), (ii), we get $\int_{SM}\Delta^s\varphi\,dm(x)\,d\nu_x = 0$. This gives another proof of the fact that $\int_{SM}\cdot dm(x)d\nu_x$ is a harmonic measure of the W^s-foliation.

Theorem 2.3. *Let M be a compact manifold of negative curvature. Then the following properties are equivalent*:

(a) $w^s = m$.
(b) $w^s = u$.
(c) $w^s = v$.
(d) w^s *is* g^t*-invariant.*
(e) *Horospheres in* $\widetilde{M}$ *have constant mean curvature.*

Proof. See [26].

A similar result can be stated for the unique harmonic measure w^u of the weak stable foliation.

3. Integral formulas for topological entropy

We continue to use the symbols and notation of §2. Let w^{ss} be the unique harmonic measure of the strong stable foliation W^{ss} of the geodesic flow. By Theorem 3.2, w^{ss} is a limit of the average measure on balls $B_x^{ss}(R)$ in $W^{ss}(x)$:

$$\frac{1}{m^{ss}B_x^{ss}(R)}\int_{B_x^{ss}(R)} \bullet\, dm^{ss}(y) \longrightarrow w^{ss} \quad (\text{as } R\to\infty).$$

On the other hand, we know that horospheres in $\widetilde{M}$ can be approximated by geodesic spheres, $H(x, \xi) = \lim_{t\to\infty} S_{v(t)}(t)$, where $v(t)$ is the geodesic in $\widetilde{M}$ satisfying $v(0) = x$ and $v(\infty) = \xi$. Thus the harmonic measure is the weak limit of the averaged measures on geodesic spheres.

To be more specific, let φ be a continuous function on SM, and x any point on $\widetilde{M}$. We define a function φ_x on $\widetilde{M}$ by $\varphi_x(y) = \varphi(v(y))$ where $v(y) \in S\widetilde{M}$ is the unique unit vector such that $v(y)(0) = y$ and $v(y)(t) = x$.

For any $\epsilon > 0$, by Theorem 3.2, there exists $R_1 > 0$ such that

$$\left|\frac{1}{m^{ss}B_v^{ss}(R)}\int_{B_v^{ss}(R)} \varphi(w)dm^{ss}(w) - \int_{SM}\varphi\, dw^{ss}\right| < \epsilon$$

for all $R \geq R_1$ and $v \in SM$. According to the estimates in [8], for a fixed $R_0 > 0$ there exists $R_2 > R_1$ such that for all $R \geq R_2$ and $y \in S_x(R)$, where $S_x(R)$ is the geodesic sphere in $\widetilde{M}$, we have

$$\left|\frac{1}{\mathrm{vol}D(y, R_0)}\int_{D(y, R_0)} \varphi_x(z)\, dz - \frac{1}{m^{ss}B_{v(y)}^{ss}(R_0)}\int_{B_{v(y)}^{ss}(R_0)} \varphi_x(z)\, dz\right| < \epsilon,$$

where $D(y, R_0)$ is the ball in $S_x(R)$ of radius R_0, and dz is the volume element of the induced Riemannian metric on $S_x(R)$.

But since $|\varphi(w) - \varphi_x(\pi(w))| \underset{R\to\infty}{\longrightarrow} 0$ uniformly for all $y \in S_x(R)$ and $w \in B_{v(y)}^{ss}(R_0)$, we can assume R_2 so large that for all $R \geq R_2$ and $y \in S_x(R)$,

$$\left|\frac{1}{m^{ss}B_{v(y)}^{ss}(R_0)}\int_{B_{v(y)}^{ss}(R_0)} (\varphi(w) - \varphi_x(\pi(w)))\, dm^{ss}(w)\right| < \epsilon.$$

So for all $R \geq R_2$,

$$\left| \frac{1}{S(x, R)} \int_{S_x(R)} \varphi_x(y)\, dy - \int_{SM} \varphi\, dw^{ss} \right| < 3\epsilon,$$

where we denote the volume of $S_x(R)$ by $S(x, R)$.

Proposition 3.1.

$$\frac{1}{S(x, R)} \int_{S_x(R)} \varphi_x(y)\, dy \overset{R\to\infty}{\longrightarrow} \int_{SM} \varphi\, dw^{ss}$$

uniformly on $\widetilde{M}$ *for any continuous function* φ *on* SM.

Now we are ready to prove

Theorem 3.2. *Let* $R(x)$ *be the scalar curvature at* x *in a closed negatively curved Riemannian manifold* M. *Let* $R^H(v)$ *be the scalar curvature of the horospheres* $H(v)$, *and let* $\mathrm{Ric}(v)$ *be the Ricci curvature of* v. *Then the topological entropy* h *of the geodesic flow satisfies*

1° $h = \int_{SM} \mathrm{tr}U(v)\, dw^{ss}(v)$,

2° $h^2 = \int_{SM} (R^H(v) - R(\pi(v)) + \mathrm{Ric}(v))\, dw^{ss}(v)$.

Proof. Margulis ([18]) proved that for any closed Riemannian manifold M of negative curvature,

$$(*) \qquad \lim_{R\to\infty} \frac{S(x, R)}{e^{hR}} = c(x)$$

for some positive continuous function c on M. Let us calculate the derivatives of the function

$$G_x(R) = \frac{S(x, R)}{e^{hR}} = \frac{1}{e^{hR}} \int_{S_x(R)} dy.$$

Then we have

$$G_x'(R) = -hG_x(R) + \frac{1}{e^{hR}} \int_{S_x(R)} \mathrm{tr}U_R(y)\, dy,$$

$$G_x''(R) = -h^2 G_x(R) - 2hG_x'(R) + \frac{1}{e^{hR}} \int_{S_x(R)} [-\mathrm{tr}\dot{U}_R(y) + (\mathrm{tr}U_R)^2]\, dy,$$

$$\begin{aligned} G_x'''(R) = &-h^3 G_x(R) - 3h^2 G_x'(R) - 3hG_x''(R) \\ &+ \frac{1}{e^{hR}} \int_{S_x(R)} (\mathrm{tr}\ddot{U}_R - 3\mathrm{tr}\dot{U}_R \mathrm{tr}U_R(y) + (\mathrm{tr}U_R)^3)\, dy, \end{aligned}$$

where $U_R(y)$ and $\mathrm{tr}U_R(y)$ are respectively the second fundamental form and the mean curvature of $S_x(R)$ at y.

Note that $\mathrm{tr}U_R(y) \to \mathrm{tr}U(v(y))$, as $R \to \infty$, uniformly. Using Proposition 6.1 and (*), we get

$$\lim_{R\to\infty} G_x'(R) = -hc(x) + c(x)\int_{SM} \mathrm{tr}U\,dw^{ss},$$

$$\lim_{R\to\infty} G_x''(R) = -h^2c(x) - 2h\lim_{R\to\infty} G_x'(R) + c(x)\int_{SM} [-\mathrm{tr}\dot{U} + (\mathrm{tr}U)^2]\,dw^{ss},$$

$$\begin{aligned}\lim_{R\to\infty} G_x'''(R) = &-h^3c(x) - 3h^2\lim_{R\to\infty} G_x'(R) - 3h\lim_{R\to\infty} G_x''(R)\\ &+ c(x)\int_{SM} [\mathrm{tr}\ddot{U} - 3\mathrm{tr}U\mathrm{tr}\dot{U} + (\mathrm{tr}U)^3]\,dw^{ss}.\end{aligned}$$

But since $\lim_{R\to\infty} G_x''(R)$ is bounded and $\lim_{R\to\infty} G_x'''(R)$ exists, we must have

$$\lim_{R\to\infty} G_x'(R) = \lim_{R\to\infty} G_x''(R) = \lim_{R\to\infty} G_x'''(R) = 0.$$

Thus

(i) $h = \int_{SM} \mathrm{tr}\,U\,dw^{ss}$,

(ii) $h^2 = \int_{SM}[-\mathrm{tr}\dot{U} + (\mathrm{tr}U)^2]\,dw^{ss}$.

Let us recall some geometry of submanifolds. If we denote by K^H the Gaussian curvature of $H(v)$ with respect to the induced Riemannian metric, then for any two orthonomal vectors X, Y in $T_{\pi(v)}H(v)$, the Gauss equation tells us

$$\begin{aligned}K^H(X, Y) = K(X, Y) &+ \langle U(v)X, X\rangle\langle U(v)Y, Y\rangle\\ &- \langle U(v)X, Y\rangle\langle X, U(v)Y\rangle,\end{aligned}$$

where U_v is a positive symmetric operator. Let $e_1, \cdots, e_{n-1}$ be its unit eigenvectors with eigenvalue $\lambda_1, \cdots, \lambda_{n-1}$. Then $K^H(e_i, e_j) = K(e_i, e_j) + \lambda_i\lambda_j$, and $\sum_{i,j} K^H(e_i, e_j) = \sum_{i,j}(K(e_i, e_j) + \lambda_i\lambda_j)$. Thus $R^H(v) = R(\pi(v)) + (\mathrm{tr}\,U)^2 - \mathrm{tr}\,U^2 - 2\mathrm{Ric}(v)$.

Remember that U satisfies the Ricatti equation $-\dot{U}+U^2+S = 0$ where $S(v)X = R(X, v)v$, R being the curvature tensor. So $\mathrm{tr}\,S(v) = \mathrm{Ric}(v)$ and $R^H(v) = R(\pi(v)) + (\mathrm{tr}\,U)^2 - \mathrm{tr}\,\dot{U} - \mathrm{Ric}(v)$. Combining this with (ii) yields

$$h^2 = \int [R^H(v) - R(\pi(v)) + \mathrm{Ric}(v)]\,dw^{ss}.$$

Remark. By our proof, there is another integral formula for the entropy:

$$h^3 = \int [\mathrm{tr}\ddot{U} + 3\,\mathrm{tr}\dot{U}\mathrm{tr}\,U + (\mathrm{tr}\,U)^3]\,dw^{ss}.$$

Actually, one can get a family of integral formulas for h^n in terms of a polynomial combination of $\operatorname{tr} U$ and its derivatives.

Corollary 3.3. *For a 3-dimensional closed Riemannian manifold of negative curvature*

$$h^2 = \int_{SM} (\operatorname{Ric}(v) - R(\pi(v)))\, dw^{ss}(v).$$

Proof. By A. Connes' Gauss-Bonnet theorem ([3]) for two-dimensional foliation (see also [5])

$$\beta_0 - \beta_1 + \beta_2 = (2\pi)^{-1} \int k(x)\, d\mu(x),$$

where μ is a completely invariant measure of the foliation, $k(x)$ denotes the Gaussian curvature function of the leaves, and $\beta_0, \beta_1, \beta_2$ are the "average Betti numbers" of the leaves relative to μ. Note that in our case, the W^{ss}-leaves have polynomial growth, each of which is conformally equivalent to and diffeomorphic to the Euclidean plane. Thus

$$\beta_0 - \beta_1 + \beta_2 = 0.$$

Now Corollary 3.3 follows from the fact that in dimension 3, $R^H(v) = 2K^H(\pi(v))$ and formula 2° in Theorem 2.2.

4. Margulis' function and applications

4.1. Patterson-Sullivan measure. We continue to use the symbols and notation of §3. In [14], Ledrappier constructed a family of finite measures $\{\mu_x\}_{x\in\widetilde{M}}$ on the sphere at infinity, satisfying the following property:

$$(*) \qquad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{x,\xi}(y)},$$

and called them the Bowen-Margulis measures, because $P_x^*(\mu_x)$ and $P|^*_{W^{su}(x)}(\mu^{su})$ are in the same measure class (recall that $P : S\widetilde{M} \to \partial\widetilde{M}$ is the canonical projection, and μ^{su} is the Margulis measure on $W^{su}(x)$).

To be more specific, let us recall Ledrappier's construction. Take a small open subset A of S_xM, and consider the following transversal T of the W^{ss}-foliation:

$$T = \bigcup_{-\delta\le t\le\delta} g_t A$$

for δ small enough. By the unique ergodicity of the W^{ss}-foliation, one can obtain a measure μ_T on T by sliding along W^{ss} leaves the Margulis measure $d\mu^u = d\mu^{su} dt$ on $W^u(x)$ which satisfies

$$d\mu_T = e^{ht}\, dt\, d\mu_A$$

for some measure μ_A on A; μ_A is exactly the measure $\mu_x|_A$ (up to a scalar constant). By the unique ergodicity of the W^{ss}-foliation and the reversibility of Ledrappier's construction, it is easy to see that any family $\{\tau_x\}_{x\in\widetilde{M}}$ of finite measures on $\partial\widetilde{M}$ satisfying (*) must coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ up to a scalar constant. For example:

(1) The Patterson-Sullivan measure [23]: Fix a point $x \in \widetilde{M}$ and consider the Poincaré series

$$g_s(y, y) = \sum_{\sigma\in\Gamma} e^{-sd(y, \sigma y)},$$

where Γ is the fundamental group of M. It converges for $s > h$ and diverges for $s < h$. Now consider the family of measures

$$\mu_x(s) = \frac{1}{g_s(y, y)} \sum_{\sigma\in\Gamma} e^{-sd(x, \sigma y)} \delta(\sigma y),$$

where $\delta(\sigma y)$ is the unit Dirac mass at σy. Let $\widetilde{\mu}_x$ be a weak limit of the family $\{\mu_x(s)\}$ as $s \to h$, then it is easy to see that

(a) $\tilde{\mu}_x$ is defined on $\partial\widetilde{M}$,

(b) $\frac{d\tilde{\mu}_y}{d\tilde{\mu}_x}(\xi) = e^{-h\rho_{x,\xi}(y)}$. By the above remark, they coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ up to a scalar constant.

(2) (Idea comes from a comment by A. Katok). Via the canonical projection $S_x(R) \to \partial\widetilde{M}$, $y \to v_y(\infty)$, where v_y is the unit normal vector of the geodesic sphere $S_x(R)$ at y, we have a sequence of finite measures defined on $\partial\widetilde{M}$:

$$\mu_x(R) = \frac{1}{e^{hR}} \int_{S_x(R)} \bullet dy.$$

By the Margulis asymptotic formula, it is easy to see that any weak limit $\widetilde{\mu}_x$ as $R \to \infty$ satisfies

(a) $\widetilde{\mu}_x$ is a measure on $\partial\widetilde{M}$ with $\widetilde{\mu}_x(\partial\widetilde{M}) = c(x)$,

(b) $\frac{d\tilde{\mu}_y}{d\tilde{\mu}_x}(\xi) = e^{-h\rho_{x,\xi}(y)}$

Thus they also coincide with $\{\mu_x\}_{x\in\widetilde{M}}$ up to a scalar constant.

Proposition 4.1. *If we denote the normalized Riemannian volume on M by $dm(x)$, by $\widetilde{\mu}_x$ and the normalized Patterson-Sullivan measure. Then* *the unique harmonic measure* w^{ss} *of the* W^{ss}*-foliation can be described as*

$$C\int_{SM} \bullet dw^{ss} = \int_M c(x)\, dm(x) \left(\int_{S_xM} \bullet d\tilde{\mu}_x(v)\right)$$

for some constant C $(C = \int_M c(x)\, dm(x))$.

Proof. Let $\{\mu_x\}_{x\in\widetilde{M}}$ be the family of Ledrappier-Patterson-Sullivan measures. Then

$$\frac{d\mu_y}{d\mu_x}(\xi) = e^{-h\rho_{x,\xi}(y)}.$$

(i) Let X be the geodesic spray. Then for any function f of class C^1_{ss}, we have

$$\begin{aligned} 0 &= \int_M dm(x)\mathrm{div}\left(\int_{S_xM} fX\, d\mu_x\right) \\ &= \int_M dm(x)\mathrm{div}|_{y=x}\left(\int_{S_xM} (fX)\, e^{-h\rho_{x,\xi}(y)} d\mu_x\right) \\ &= \int_{SM} [\dot{f} + (h - \mathrm{tr}\, U)f]\, dm(x)\, d\mu_x, \end{aligned}$$

where we define $\int_{SM} \varphi\, dm(x) d\mu_x \stackrel{\text{def}}{=} \int_M dm(x)(\int_{S_xM} \varphi(x, \xi)\, d\mu_x(\xi))$.

(ii) For any function f of class C^2_{ss},

$$\begin{aligned} 0 &= \int_M dm(x)\Delta\left(\int_{S_xM} f\, d\mu_x\right) \\ &= \int_M dm(x)\Delta|_{y=x}\left(\int_{S_xM} fe^{-h\rho_{x,\xi}(y)} d\mu_x\right) \\ &= \int_{SM} (\Delta^s f + hf(h - \mathrm{tr}U) + 2h\dot{f})\, dm(x)\, d\mu_x \\ &= \int_{SM} [\Delta^{ss} f + (\ddot{f} + (h - \mathrm{tr}U)\dot{f}) + h(\dot{f} + (h - \mathrm{tr}U)f)]\, dm(x)\, d\mu_x. \end{aligned}$$

Combining (i) and (ii) yields $\int_{SM} \Delta^{ss} f\, dm(x)\, d\mu_x = 0$. By the uniqueness of the W^{ss}-harmonic measure, $dw^{ss} = dm(x) d\mu_x$, up to normalization. Using Proposition 6.1 and Margulis' asymptotic formula, one can see that

$$dw^{ss}(x, \xi) = \frac{c(x)}{\int_M c(x)\, dm(x)}\, dm(x)\, d\tilde{\mu}_x(\xi),$$

where $\widetilde{\mu}_x$ is the normalized Ledrappier-Patterson-Sullivan measure.

4.2. Margulis function. The following theorem implies that for compact manifolds with negative curvature, the function $c(x)$ is almost always not a constant function.

Theorem 4.2.

(i) $c(x)$ *is smooth, if the Riemannian metric is smooth.*

(ii) *If* $c(x) \equiv C$, *then for each* $x \in \widetilde{M}$,

$$h = \int_{\partial \widetilde{M}} \operatorname{tr} U(x, \xi)\, d\widetilde{\mu}_x(\xi).$$

Proof. Note that

$$\frac{d\widetilde{\mu}_y}{d\widetilde{\mu}_x}(\xi) = \frac{c(x)}{c(y)} e^{-h\rho_{x,\xi}(y)}.$$

Then (i) follows from the fact that $c(y) = c(x) \int_{\partial \widetilde{M}} e^{-h\rho_{x,\xi}(y)}\, d\widetilde{\mu}_x(\xi)$. If $c(x) \equiv$ const., then $\int e^{-h\rho_{x,\xi}(y)}\, d\widetilde{\mu}_x(\xi) \equiv 1$. Taking the Laplacian on both sides yields

$$\int h(h - \operatorname{tr} U) e^{-h\rho_{x,\xi}(y)} d\widetilde{\mu}_x(\xi) = 0.$$

Thus $h = \int_{\partial \widetilde{M}} \operatorname{tr} U(x, \xi)\, d\widetilde{\mu}_x(\xi)$.

Theorem 4.3. *If* $\dim M = 2$ *and* $c(x) \equiv$ const., *then* M *has constant negative curvature.*

Proof. According to Theorem 3.2, $h^2 = \int (-\operatorname{tr} \dot{U} + (\operatorname{tr} U)^2)\, dw^{ss}$. Using the Ricatti equation $-\dot{U} + U^2 + S = 0$, and noting that $\operatorname{tr} U^2 = (\operatorname{tr} U)^2$ in dimension 2 we have

$$h^2 = \int -\operatorname{tr} S\, dw^{ss} = -\int K\, dm(x) = -2\pi E,$$

where E is the Euler characteristic of M, and K is the Gaussian curvature. Hence Theorem 4.3 follows from A. Katok's result [11] that $h^2 = -2\pi E$ if and only if M has constant negative curvature.

The following corollary measures the deviation of metrics from constant negative curvature.

Corollary 4.2. *If* $\dim M = 2$, *then*

$$h^2 = \frac{1}{C} \int_M - c(x)\, k(x)\, dm(x).$$

Due to the above facts, it makes sense to have the following conjecture.

Conjecture. *For a compact Riemannian manifold* M *of negative curvature,* $c(x) \equiv$ const. *if and only if* M *is locally symmetric.*

4.3. Margulis function and flip-invariance of Patterson-Sullivan measure. Recall that the strong unstable foliation W^{su} also has a unique harmonic measure w^{su}. By the flip map, we get

$$C\,dw^{su}(x, \xi) = c(x)\,dm(x)\,d\tilde{\mu}_x(-\xi).$$

Thus $w^{su} = w^{ss}$ if and only if $d\tilde{\mu}_x(-\xi) = d\tilde{\mu}_x(\xi)$. Ledrappier [15] proves that if $\dim M = 2$, then $w^{su} = w^{ss}$ if and only if M has constant curvature. The following result indicates that in higher dimensional case, locally symmetric spaces might be the only manifolds of negative curvature for which $w^{ss} = w^{su}$.

Corollary 4.4. *If $w^{ss} = w^{su}$, then $c(x) \equiv \text{const}$.*

Proof. Any C^2 function φ on M can be lifted to a function on SM which we denote by the same symbol. By the proof of our Proposition 7.1 or by Corollary 1 of [15], we have

$$\begin{aligned}\int_M \Delta\varphi c(x)\,dm &= C\int_{SM} \Delta^s\varphi\,dw^{ss} \\ &= C\int_{SM} (\Delta^{ss}\varphi + \ddot{\varphi} - \operatorname{tr}U\dot{\varphi})\,dw^{ss} \\ &= C\left[\int_{SM}\Delta^{ss}\varphi\,dw^{ss} + \int_{SM}(\ddot{\varphi} + (h - \operatorname{tr}U)\dot{\varphi})\,dw^{ss} - \int_{SM} h\dot{\varphi}\,dw^{ss}\right] \\ &= -h\int_M c(x)\,dm(x)\int_{\partial\tilde{M}} \dot{\varphi}(x, \xi)\,d\tilde{\mu}_x(\xi),\end{aligned}$$

where $d\tilde{\mu}_x(\xi) = d\tilde{\mu}_x(-\xi)$ and $\dot{\varphi}(x, \xi) = -\dot{\varphi}(x, -\xi)$. Thus

$$\int_M \Delta\varphi\; c(x)\;dm(x) = 0$$

for all C^2 function φ on M. It follows that $c(x) \equiv \text{const}$.

4.4. An upper bound of Gromov's simplicial volume. Following [6], if $c = \sum_{i=1}^k C_i \cdot S_i$ is the decomposition of a real chain c in terms of the elementary simplices S_i, then the ℓ_1-norm of c is defined to be $\|c\| = \sum_{k=1}^k |C_i|$. For every homology class σ, one can define $\|\sigma\| = \inf\{\|c\| \mid [c] = \sigma\}$. Thus the simplicial volume of a closed manifold M is, by definition, equal to $\|[M]\|$, where $[M]$ is the fundamental class of M. Roughly speaking, simplicial volume is the minimal number of simplices needed to triangulate the fundamental classes of M. From now on we denote it by $\|M\|$. For a closed manifold M of negative curvature, Gromov proved that (see [6]) there exists a constant $C = c(n)$ which depends only on $n = \dim M$, such that

$$(**) \qquad \|M\| \leq Ch^n V(M),$$

where $V(M)$ is the volume of M, and h is the topological entropy of the geodesic flow on M.

If the Ricci curvatures of M are bounded from below, i.e., $\mathrm{Ric}(M) \geq -r_0^2$, then by the Bishop volume comparison theorem, it is easy to see that $h \leq \sqrt{n-1}\, r_0$. It follows that

$$\|M\| \leq C \cdot r_0^n V(M).$$

Therefore the simplicial volume is controlled by the lower bound of Ricci curvature. Gromov raised the following conjecture.

Conjecture (Gromov [7, p. 117]). *Every n-dimensional closed manifold M with $R(M) \geq -\sigma^2$ satisfies $\|M\| \leq C(n)\sigma^n V(M)$.*

Here $R(M)$ denotes the scalar curvature of M. Actually, Gromov made a stronger conjecture that

$$\|M\| \leq C(n) \int_M |R^-(x)|^{n/2}\, dx,$$

where $R^-(x) = \min(0, R(x))$, and dx is the Riemannian volume. As is pointed out in [7], one does not know if every hyperbolic 3-manifold admits a sequence of metrics such that $\int_M |R|^{3/2} \to 0$, even if one insists on $K < 0$ for these metrics.

By Theorem 3.2 and (**) we have

Theorem 4.5. *Let M be a closed Riemannian manifold of negative sectional curvature, and $c(x)$ be the Margulis function. Then*

$$\|M\| \leq C(n)V(M) \cdot \left[\frac{\int_M (H(x) + |R(x)|)c(x)\, dx}{\int_M c(x)\, dx}\right]^n,$$

where $H(x)$ is the suppremume of the scalar curvatures at x of the horospheres through x. In particular, if $\dim M = 3$, then

$$\|M\| \leq C \cdot V(M) \cdot \left[\frac{\int_M (H(x) + |R(x)|)c(x)\, dx}{\int_M c(x)\, dx}\right]^3 \leq C \cdot V(M)\sigma^3,$$

where $\sigma = \sup_{x\in M} |R(x)|$.

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measure μ_x constructed by Ledrappier [14]. After finishing most (§§1–4) of this paper, I received a preprint [15] from Professor Ledrappier in which he proves the unique ergodicity for Anosov foliations of geodesic flows, Proposition 3.1 as well as formula 1° of Theorem 3.2. In an independent work [13], Professor Knieper also proves, among other things, formula 2° of Theorem 3.2. I owe particular gratitude to the referee for many suggestions.

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